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On (α, β, γ)-derivations of Lie algebras and corresponding invariant functions

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Abstract

We consider finite dimensional complex Lie algebras. We generalize the concept of Lie derivations via certain complex parameters and obtain various Lie and Jordan operator algebras as well as two one-parametric sets of linear operators. Using these parametric sets, we introduce complex functions with a fundamental property — invariance under Lie isomorphisms. One of these basis-independent functions represents a complete set of invariant(s) for three-dimensional Lie algebras. We present also its application to physically motivated examples in dimension 8.
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1. Introduction

The theory of finite dimensional complex Lie algebras is an important part of Lie theory. It has several applications to physics and connections to other parts of mathematics. With an increasing amount of theory and applications concerning Lie algebras of various dimensions, it is becoming necessary to ascertain applicable tools for handling them. The miscellaneous characteristics of Lie algebras constitute such tools and have also found applications: *Casimir operators* [1], *derived*, *lower central* and *upper central sequences*, *Lie algebra of derivations*, *radical*, *nilradical*, *ideals*, *subalgebras* [7,14] and recently *megaideals*, [13]. These characteristics are particularly crucial when considering possible affinities among Lie algebras.

Physically motivated relations between two Lie algebras, namely *contractions* and *deformations*, have been extensively studied; see e.g. [3,9]. When investigating these kinds of relations in dimensions higher than 5, one can encounter insurmountable difficulties. Firstly, aside from the *semisimple* ones, Lie algebras are completely classified only up to dimension 5 and the *nilpotent* ones up to dimension 6. In higher dimensions, only special types, such as *rigid* Lie algebras [4] or Lie algebras with fixed structure of the nilradical, are classified [15] (for a detailed survey of

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classification results in lower dimensions see e.g. [13] and references therein). Secondly, if all available characteristics of two results of contraction/deformation are the same then one cannot distinguish them at all. This often occurs when the result of a contraction is class of Lie algebras with one parameter or more.

The aim of this article is to partially overcome these obstacles and to add new objects to the existing set of invariants. Number of Casimir operators, dimensions of radical, nilradical, lower central sequences, etc., are all invariant or – equivalently – basis independent. However, in this article we pursue different kinds of basis-independent characteristics – certain complex functions. These invariant functions, which arise from the concept of so called (α, β, γ)-derivations, then represent a very powerful tool for the description of Lie algebras, very effective and essential when dealing with their parametric continuum.

In Section 2, we generalize the concept of derivation of a Lie algebra; we introduce (α, β, γ)-derivations and show their pertinent properties. All possible intersections of spaces containing these derivations are investigated. Examples of low dimensional Lie algebras are presented.

In Section 3, we introduce two invariant functions corresponding to (α, β, γ)-derivations. We demonstrate, on all three-dimensional complex Lie algebras and on physically motivated examples in dimension 8, how these functions effectively enlarge the set of 'classical' invariants.

In Section 4, we briefly review other generalizations of derivations, make a note on a computation of (α, β, γ)-derivations and make further comments.

2. (α, β, γ)-derivations

In this article let $\mathcal{L}$ denote the finite dimensional Lie algebra over the field of complex numbers $\mathbb{C}$ and $\mathrm{End}(\mathcal{L})$ the associative algebra of all linear operators on the vector space $\mathcal{L}$. The space $\mathrm{End}(\mathcal{L})$, endowed with standard Lie commutator $[A, B] = AB - BA$, is denoted as usual by $\mathrm{gl}(\mathcal{L})$ and the space $\mathrm{End}(\mathcal{L})$, endowed with Jordan product $A \circ B = \frac{1}{2}(AB + BA)$, is denoted by $\mathrm{jor}(\mathcal{L})$. In this way, any subalgebra of $\mathrm{End}(\mathcal{L})$ forms also a subalgebra of $\mathrm{gl}(\mathcal{L})$ and $\mathrm{jor}(\mathcal{L})$. We also adopt notation for the *center* $C(\mathcal{L})$ and for the *derived algebra* $\mathcal{L}^2 = [\mathcal{L}, \mathcal{L}]$.

2.1. Properties and structure of (α, β, γ)-derivations

Recall that a *derivation* of $\mathcal{L}$ is a linear operator $A \in \mathrm{End}(\mathcal{L})$ such that for all $x, y \in \mathcal{L}$

$$A[x, y] = [Ax, y] + [x, Ay] \tag{1}$$

and the set of all these derivations, denoted by $\mathrm{der}(\mathcal{L})$, forms a Lie algebra of derivations. Several non-equivalent ways of generalizing this definition have been recently studied [2,8,5]. However, we will bring forward another type of generalization.

We call a linear operator $A \in \mathrm{End}(\mathcal{L})$ an (α, β, γ)-**derivation** of $\mathcal{L}$ if there exist $\alpha, \beta, \gamma \in \mathbb{C}$ such that for all $x, y \in \mathcal{L}$

$$\alpha A[x, y] = \beta[Ax, y] + \gamma[x, Ay]. \tag{2}$$

For given $\alpha, \beta, \gamma \in \mathbb{C}$ we denote the set of all (α, β, γ)-derivations as $\mathcal{D}(\alpha, \beta, \gamma)$, i.e.

$$\mathcal{D}(\alpha, \beta, \gamma) = \{A \in \mathrm{End}(\mathcal{L}) \mid \alpha A[x, y] = \beta[Ax, y] + \gamma[x, Ay],\ \forall x, y \in \mathcal{L}\}. \tag{3}$$

Let us focus on this set and show some its properties. It is clear that $\mathcal{D}(\alpha, \beta, \gamma)$ is a linear subspace of $\mathrm{End}(\mathcal{L})$ and it follows immediately from (2) that for any $\varepsilon \in \mathbb{C} \setminus \{0\}$ it holds that

$$\mathcal{D}(\alpha, \beta, \gamma) = \mathcal{D}(\alpha\varepsilon, \beta\varepsilon, \gamma\varepsilon) = \mathcal{D}(\alpha, \gamma, \beta). \tag{4}$$

Furthermore, we have the following important property:

Lemma 2.1. *For any $\alpha, \beta, \gamma \in \mathbb{C}$,*

$$\mathcal{D}(\alpha, \beta, \gamma) = \mathcal{D}(0, \beta - \gamma, \gamma - \beta) \cap \mathcal{D}(2\alpha, \beta + \gamma, \beta + \gamma). \tag{5}$$

Proof. Suppose any $\alpha, \beta, \gamma \in \mathbb{C}$ are given. Then for $A \in \mathcal{D}(\alpha, \beta, \gamma)$ and arbitrary $x, y \in \mathcal{L}$ we have

$$\begin{aligned} \alpha A[x, y] &= \beta[Ax, y] + \gamma[x, Ay] \\ \alpha A[y, x] &= \beta[Ay, x] + \gamma[y, Ax]. \end{aligned} \tag{6}$$

By summing and subtracting Eq. (6) we obtain

$$\begin{aligned} 0 &= (\beta - \gamma)([Ax, y] - [x, Ay]) \\ 2\alpha A[x, y] &= (\beta + \gamma)([Ax, y] + [x, Ay]) \end{aligned} \tag{7}$$

and thus $\mathcal{D}(\alpha, \beta, \gamma) \subset \mathcal{D}(0, \beta - \gamma, \gamma - \beta) \cap \mathcal{D}(2\alpha, \beta + \gamma, \beta + \gamma)$. Similarly, starting with Eqs. (7) we obtain Eqs. (6) and the remaining inclusion is proven. □

Further, we proceed to formulate the theorem which reveals the structure of the spaces $\mathcal{D}(\alpha, \beta, \gamma)$; the three original parameters are in fact reduced to only one.

Theorem 2.2. *For any $\alpha, \beta, \gamma \in \mathbb{C}$ there exists $\delta \in \mathbb{C}$ such that the subspace $\mathcal{D}(\alpha, \beta, \gamma) \subset \mathrm{End}(\mathcal{L})$ is equal to some of the four following subspaces:*

(1) $\mathcal{D}(\delta, 0, 0)$.
(2) $\mathcal{D}(\delta, 1, -1)$.
(3) $\mathcal{D}(\delta, 1, 0)$.
(4) $\mathcal{D}(\delta, 1, 1)$.

Proof. (1) Suppose $\beta + \gamma = 0$. Then either $\beta = \gamma = 0$ or $\beta = -\gamma \neq 0$.
(a) For $\beta = \gamma = 0$ we have
$$\mathcal{D}(\alpha, \beta, \gamma) = \mathcal{D}(\alpha, 0, 0).$$
(b) For $\beta = -\gamma \neq 0$ we have according to (4) and (5)
$$\mathcal{D}(\alpha, \beta, \gamma) = \mathcal{D}(0, \beta - \gamma, \gamma - \beta) \cap \mathcal{D}(2\alpha, 0, 0) = \mathcal{D}(0, 1, -1) \cap \mathcal{D}(\alpha, 0, 0).$$
On the other hand it holds that
$$\mathcal{D}(\alpha, 1, -1) = \mathcal{D}(0, 2, -2) \cap \mathcal{D}(2\alpha, 0, 0) = \mathcal{D}(0, 1, -1) \cap \mathcal{D}(\alpha, 0, 0)$$
and therefore
$$\mathcal{D}(\alpha, \beta, \gamma) = \mathcal{D}(\alpha, 1, -1).$$
(2) Suppose $\beta + \gamma \neq 0$. Then either $\beta - \gamma \neq 0$ or $\beta = \gamma \neq 0$.
(a) For $\beta - \gamma \neq 0$ we have
$$\mathcal{D}(\alpha, \beta, \gamma) = \mathcal{D}(0, \beta - \gamma, \gamma - \beta) \cap \mathcal{D}(2\alpha, \beta + \gamma, \beta + \gamma) = \mathcal{D}(0, 1, -1) \cap \mathcal{D}\left(\frac{2\alpha}{\beta + \gamma}, 1, 1\right)$$
and this is according to (5) equal to $\mathcal{D}(\frac{\alpha}{\beta+\gamma}, 1, 0)$, i.e.
$$\mathcal{D}(\alpha, \beta, \gamma) = \mathcal{D}\left(\frac{\alpha}{\beta + \gamma}, 1, 0\right).$$
(b) For $\beta = \gamma \neq 0$ we have
$$\mathcal{D}(\alpha, \beta, \gamma) = \mathcal{D}\left(\frac{\alpha}{\beta}, 1, 1\right). \quad \square$$

Now we will discuss in detail the possible outcome of Theorem 2.2 which depends on the value of the parameter $\delta \in \mathbb{C}$.

(1) $\mathcal{D}(\delta, 0, 0)$:
(a) For $\delta = 0$ we trivially get $\mathcal{D}(0, 0, 0) = \mathrm{End}(\mathcal{L})$.
(b) For $\delta \neq 0$ the space $\mathcal{D}(1, 0, 0)$ is an associative subalgebra of $\mathrm{End}(\mathcal{L})$, which maps derived algebra $\mathcal{L}^2 = [\mathcal{L}, \mathcal{L}]$ to the zero vector:
$$\mathcal{D}(1, 0, 0) = \{A \in \mathrm{End}(\mathcal{L}) \mid A(\mathcal{L}^2) = 0\},$$
and therefore its dimension is as follows:
$$\dim \mathcal{D}(1, 0, 0) = \mathrm{codim}\, \mathcal{L}^2 \dim \mathcal{L}.$$

(2) $\mathcal{D}(\delta, 1, -1)$:
   (a) For $\delta = 0$ we have a Jordan algebra $\mathcal{D}(0, 1, -1) \subset \mathrm{jor}(\mathcal{L})$,
   $$\mathcal{D}(0, 1, -1) = \{A \in \mathrm{End}(\mathcal{L}) \mid [Ax, y] = [x, Ay],\ \forall x, y \in \mathcal{L}\}.$$
   (b) For $\delta \neq 0$ we get Jordan algebra $\mathcal{D}(1, 1, -1) \subset \mathrm{jor}(\mathcal{L})$ as an intersection of two Jordan algebras:
   $$\mathcal{D}(\delta, 1, -1) = \mathcal{D}(0, 1, -1) \cap \mathcal{D}(\delta, 0, 0) = \mathcal{D}(0, 1, -1) \cap \mathcal{D}(1, 0, 0) = \mathcal{D}(1, 1, -1).$$

(3) $\mathcal{D}(\delta, 1, 0)$:
   (a) For $\delta = 0$ we get an associative algebra of all linear operators of the vector space $\mathcal{L}$, which maps the whole $\mathcal{L}$ into its center $C(\mathcal{L})$:
   $$\mathcal{D}(0, 1, 0) = \{A \in \mathrm{End}(\mathcal{L}) \mid A(\mathcal{L}) \subseteq C(\mathcal{L})\},$$
   and its dimension is
   $$\dim \mathcal{D}(0, 1, 0) = \dim \mathcal{L}\ \dim C(\mathcal{L}).$$
   (b) For $\delta = 1$ the space $\mathcal{D}(1, 1, 0)$ is the centralizer of adjoint representation $\mathrm{ad}(\mathcal{L})$ in $\mathrm{gl}(\mathcal{L})$.
   (c) For the remaining values of δ the space $\mathcal{D}(\delta, 1, 0)$ forms, in the general case of Lie algebra $\mathcal{L}$, only the vector subspace of $\mathrm{End}(\mathcal{L})$. Thus, we have the one-parametric set of vector spaces:
   $$\mathcal{D}(\delta, 1, 0) = \mathcal{D}(0, 1, -1) \cap \mathcal{D}(2\delta, 1, 1).$$

(4) $\mathcal{D}(\delta, 1, 1)$:
   (a) For $\delta = 0$ we have a Lie algebra
   $$\mathcal{D}(0, 1, 1) = \{A \in \mathrm{End}(\mathcal{L}) \mid [Ax, y] = -[x, Ay],\ \forall x, y \in \mathcal{L}\}.$$
   (b) For $\delta = 1$ we get the algebra of derivations of $\mathcal{L}$,
   $$\mathcal{D}(1, 1, 1) = \mathrm{der}(\mathcal{L}).$$
   (c) For the remaining values of δ the space $\mathcal{D}(\delta, 1, 1)$ forms, in the general case of Lie algebra $\mathcal{L}$, only the vector subspace of $\mathrm{End}(\mathcal{L})$.

2.2. *Intersections of the spaces $\mathcal{D}(\alpha, \beta, \gamma)$*

Various intersections of two different subspaces $\mathcal{D}(\alpha, \beta, \gamma)$ also turned out to be of interest; therefore we systematically explore all possible intersections of these spaces. However, all intersections of these spaces lead us to only two new structures. The first one

$$\begin{aligned}\mathcal{D}(1, 0, 0) \cap \mathcal{D}(0, 1, 0) &= \mathcal{D}(1, 0, 0) \cap \mathcal{D}(\delta, 1, 0) = \mathcal{D}(1, 1, -1) \cap \mathcal{D}(\delta, 1, 0)\\ &= \mathcal{D}(1, 1, -1) \cap \mathcal{D}(\delta, 1, 1) = \mathcal{D}(\delta, 1, 0) \cap \mathcal{D}(\gamma, 1, 0)_{\delta \neq \gamma}\\ &= \mathcal{D}(\delta, 1, 0) \cap \mathcal{D}(\gamma, 1, 1)_{2\delta \neq \gamma}\end{aligned} \tag{8}$$

forms an associative algebra and is contained in all spaces $\mathcal{D}(\alpha, \beta, \gamma)$. Its dimension is

$$\dim(\mathcal{D}(1, 0, 0) \cap \mathcal{D}(0, 1, 0)) = \mathrm{codim}\, \mathcal{L}^2\, \dim C(\mathcal{L}). \tag{9}$$

The second one is a new Lie algebra:

$$\mathcal{D}(1, 0, 0) \cap \mathcal{D}(0, 1, 1) = \mathcal{D}(1, 0, 0) \cap \mathcal{D}(\delta, 1, 1) = \mathcal{D}(\delta, 1, 1) \cap \mathcal{D}(\gamma, 1, 1)_{\delta \neq \gamma}. \tag{10}$$

Other intersections lead to structures which we already have:

$$\begin{aligned}&\mathcal{D}(1, 1, -1) = \mathcal{D}(1, 0, 0) \cap \mathcal{D}(0, 1, -1) = D(1, 0, 0) \cap D(1, 1, -1) = D(0, 1, -1) \cap D(1, 1, -1)\\ &\mathcal{D}(\delta, 1, 0) = \mathcal{D}(0, 1, -1) \cap \mathcal{D}(\delta, 1, 0) = \mathcal{D}(0, 1, -1) \cap \mathcal{D}(2\delta, 1, 1) = D(\delta, 1, 0) \cap D(2\delta, 1, 1).\end{aligned} \tag{11}$$

For completeness we state that the space $\mathcal{D}(1, 0, 0) \cap \mathcal{D}(0, 1, 0)$ forms an ideal in $\mathcal{D}(1, 1, 0)$, $\mathcal{D}(1, 1, 1)$ and in $\mathcal{D}(1, 1, -1)$; the space $\mathcal{D}(1, 0, 0) \cap \mathcal{D}(0, 1, 1)$ is an ideal in $\mathcal{D}(1, 1, 1)$; the space $\mathcal{D}(0, 1, 0)$ forms an ideal in $\mathcal{D}(0, 1, 1)$ and in $\mathcal{D}(0, 1, -1)$.

The structure of algebras $\mathcal{D}(1, 0, 0)$, $\mathcal{D}(0, 1, 0)$ and their intersection depends only on the dimensions of the center and the derived algebra of $\mathcal{L}$. If $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are the Lie algebras of the same dimension then the same dimension of their centers implies $\mathcal{D}(0, 1, 0) \cong \widetilde{\mathcal{D}}(0, 1, 0)$; the coinciding dimension of the derived algebra implies

$\mathcal{D}(1,0,0) \cong \widetilde{\mathcal{D}}(1,0,0)$. Moreover, if $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are both indecomposable and dimensions of the centers coincide, as well as the dimensions of derived algebras, then it holds that:

$$\mathcal{D}(0,1,0) \cap \mathcal{D}(1,0,0) \cong \widetilde{\mathcal{D}}(0,1,0) \cap \widetilde{\mathcal{D}}(1,0,0). \tag{12}$$

2.3. Examples of (α, β, γ)-derivations

We present as illustration some examples of (α, β, γ)-derivations for lower dimensional Lie algebras. Note especially the form of the one-parametric subspace $\mathcal{D}(\delta, 1, 1)$, as it is significant later on.

Example 2.1. Consider a two-dimensional Lie algebra $\mathcal{L}_2$ with a basis $\{e_1, e_2\}$ and its only non-zero commutation relation: $[e_1, e_2] = e_2$.

- $\mathcal{D}(1,1,1) = \mathrm{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cong \mathcal{L}_2.$
- $\mathcal{D}(0,1,1) = \mathrm{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \cong sl(2, \mathbb{C}).$
- $\mathcal{D}(1,1,0) = \mathcal{D}(0,1,-1) = \mathrm{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$
- $\mathcal{D}(1,0,0) \cap \mathcal{D}(0,1,1) = \mathrm{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$
- $\mathcal{D}(\delta,1,0) = \{0\}$ for $\delta \neq 1$.
- $\mathcal{D}(\delta,1,1) = \mathrm{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \delta - 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ for $\delta \neq 0$.
- $\mathcal{D}(0,1,0) = \mathcal{D}(1,0,0) \cap \mathcal{D}(0,1,0) = \{0\}.$
- $\mathcal{D}(1,0,0) = \mathrm{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \cong \mathcal{L}_2.$
- $\mathcal{D}(1,1,-1) = \{0\}.$

Example 2.2. Consider a simple Lie algebra of the lowest dimension: $sl(2, \mathbb{C})$.

- $\mathcal{D}(1,1,1) \cong sl(2, \mathbb{C}).$
- $\mathcal{D}(0,1,1) = \mathcal{D}(0,1,0) = \mathcal{D}(1,0,0) = \mathcal{D}(1,1,-1) = \{0\}.$
- $\mathcal{D}(1,1,0) = \mathcal{D}(0,1,-1) = \mathrm{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$
- $\mathcal{D}(\delta,1,0) = \{0\}$ for $\delta \neq 1$.
- $\mathcal{D}(\delta,1,1) = \{0\}$ for $\delta \neq \pm 1, 2$.
- $\mathcal{D}(2,1,1) = \mathrm{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$
- $\mathcal{D}(-1,1,1) = \mathrm{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}.$

3. Invariant functions

3.1. Definition and properties of functions ψ and ϕ

In this section we define complex functions with a pertinent property — invariance under Lie isomorphisms. Suppose we have an arbitrary non-singular linear mapping σ and this mapping represents an isomorphism between two Lie algebras, say $\mathcal{L}$ and $\widetilde{\mathcal{L}}$. That means that for $\sigma : \mathcal{L} \to \widetilde{\mathcal{L}}$ and for all $x, y \in \widetilde{\mathcal{L}}$

$$[x, y]_{\widetilde{\mathcal{L}}} = \sigma[\sigma^{-1}x, \sigma^{-1}y]_{\mathcal{L}}.$$

By rewriting definition relation (2) we have for $A \in \mathcal{D}(\alpha, \beta, \gamma)$

$$\alpha A[\sigma^{-1}x, \sigma^{-1}y]_{\mathcal{L}} = \beta[A\sigma^{-1}x, \sigma^{-1}y]_{\mathcal{L}} + \gamma[\sigma^{-1}x, A\sigma^{-1}y]_{\mathcal{L}}.$$

Applying the mapping σ to this equation and taking into account that α, β, γ are in $\mathbb{C}$ then

$$\alpha\sigma A\sigma^{-1}[x, y]_{\widetilde{\mathcal{L}}} = \beta[\sigma A\sigma^{-1}x, y]_{\widetilde{\mathcal{L}}} + \gamma[x, \sigma A\sigma^{-1}y]_{\widetilde{\mathcal{L}}}, \tag{13}$$

i.e. $\sigma A\sigma^{-1} \in \widetilde{\mathcal{D}}(\alpha, \beta, \gamma)$. Thus, we easily arrive at the crucial result which we sum up as follows:

Proposition 3.1. *Let $\sigma : \mathcal{L} \to \widetilde{\mathcal{L}}$ be an isomorphism of Lie algebras $\mathcal{L}$ and $\widetilde{\mathcal{L}}$. Then the mapping $\rho : \mathrm{End}(\mathcal{L}) \to \mathrm{End}(\widetilde{\mathcal{L}})$ defined for all $A \in \mathrm{End}(\mathcal{L})$ by $\rho(A) = \sigma A \sigma^{-1}$ is an isomorphism of associative algebras $\mathrm{End}(\mathcal{L})$ and $\mathrm{End}(\widetilde{\mathcal{L}})$. Moreover, for any $\alpha, \beta, \gamma \in \mathbb{C}$,*

$$\rho(\mathcal{D}(\alpha, \beta, \gamma)) = \widetilde{\mathcal{D}}(\alpha, \beta, \gamma)$$

holds.

Corollary 3.2. *For any $\alpha, \beta, \gamma \in \mathbb{C}$ the dimension of the vector space $\mathcal{D}(\alpha, \beta, \gamma)$ is an invariant of the Lie algebra $\mathcal{L}$.*

Indeed, the possibility of new invariants as dimensions of various $\mathcal{D}(\alpha, \beta, \gamma)$ is very promising; it seems timely to list a summary of those spaces $\mathcal{D}(\alpha, \beta, \gamma)$ whose dimensions do not depend on the dimensionality of well-known substructures of $\mathcal{L}$, such as the center $C(\mathcal{L})$ and $\mathcal{L}^2$. The outcome of Theorem 2.2 and the discussion below it, relations (10) and (12), yield:

(1) associative algebra $\mathcal{D}(1, 1, 0)$,
(2) Lie algebras $\mathcal{D}(1, 1, 1)$, $\mathcal{D}(0, 1, 1)$, $\mathcal{D}(1, 0, 0) \cap \mathcal{D}(0, 1, 1)$,
(3) Jordan algebras $\mathcal{D}(1, 1, -1)$, $\mathcal{D}(0, 1, -1)$,
(4) one-parametric sets of vector spaces $\mathcal{D}(\alpha, 1, 0)$, $\mathcal{D}(\alpha, 1, 1)$.

Since the definition of (α, β, γ)-derivations partially overlaps other generalizations, some of these sets naturally appeared already in the literature. Namely in [8], a considerable amount of theory concerning relations between $\mathcal{D}(1, 1, 0)$ and $\mathcal{D}(0, 1, -1)$ has been developed (see also Section 4). In [10], the usefulness of the invariant dimensions of the Lie structures $\mathcal{D}(1, 1, 1)$, $\mathcal{D}(0, 1, 1)$, $\mathcal{D}(1, 1, 0)$ and $\mathcal{D}(1, 0, 0) \cap \mathcal{D}(0, 1, 1)$ as well as their mutual independence has been shown. In this article we focus on one-parametric sets of vector spaces $\mathcal{D}(\alpha, 1, 0)$ and $\mathcal{D}(\alpha, 1, 1)$.

We use these one-parametric sets of vector spaces to define the invariant function of a Lie algebra $\mathcal{L}$. Functions $\psi, \phi : \mathbb{C} \longrightarrow \{0, 1, 2, \ldots, (\dim \mathcal{L})^2\}$ defined by formulas

$$\psi(\alpha) = \dim \mathcal{D}(\alpha, 1, 1) \tag{14}$$

$$\phi(\alpha) = \dim \mathcal{D}(\alpha, 1, 0) \tag{15}$$

are called **invariant functions** corresponding to (α, β, γ)-derivations of a Lie algebra $\mathcal{L}$. We observe that from the relations (8), (9) and (11) it follows that

$$\operatorname{codim} \mathcal{L}^2 \dim C(\mathcal{L}) \leq \phi(\alpha) \leq \dim \mathcal{D}(0, 1, -1)$$

$$\phi(\alpha) \leq \psi(2\alpha)$$

$$\operatorname{codim} \mathcal{L}^2 \dim C(\mathcal{L}) \leq \psi(\alpha).$$

From Proposition 3.1 it follows immediately that for two Lie algebras $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ it holds that

$$\mathcal{L} \cong \widetilde{\mathcal{L}} \Rightarrow \psi = \widetilde{\psi} \quad \text{and} \quad \phi = \widetilde{\phi}.$$

Note that sometimes in the literature, the name "invariant function" denotes a (formal) Casimir invariant; its form however depends on the choice of a basis of $\mathcal{L}$. Here by invariant functions we mean rather 'basis-independent' complex functions, such as ψ and ϕ. Since the purpose of the functions ψ a ϕ is to enlarge the set of 'classical' invariants which we list below, this terminology is well justified.

The classical method of identification of an (indecomposable) Lie algebra [14] boils down to computation of derived series $D^{k+1}(\mathcal{L}) = [D^k(\mathcal{L}), D^k(\mathcal{L})]$, $D^0(\mathcal{L}) = \mathcal{L}$, lower central series $\mathcal{L}^{k+1} = [\mathcal{L}^k, \mathcal{L}]$, $\mathcal{L}^1 = \mathcal{L}$, and upper central series

$$C^{k+1}(\mathcal{L})/C^k(\mathcal{L}) = C(\mathcal{L}/C^k(\mathcal{L})), C^1(L) = C(L).$$

The dimensions of these ideals and the dimensions of the spaces of (α, β, γ)-derivations are "numerical" invariants of the Lie algebra as is the number of formal Casimir invariants $\tau(\mathcal{L})$ [1]. These characteristics, applied to radical, nilradical and factors of *Levi decomposition* of $\mathcal{L}$, also form invariants. We adopt the following notation:

$$\begin{aligned}\mathrm{inv}(\mathcal{L}) = {} & (\dim D^k(\mathcal{L}))\ (\dim \mathcal{L}^k)\ (\dim C^k(\mathcal{L}))\ \tau(\mathcal{L}) \\ & [\dim \mathcal{D}(1, 1, 1), \dim \mathcal{D}(0, 1, 1), \dim \mathcal{D}(1, 1, 0), \\ & \dim(\mathcal{D}(1, 1, 1) \cap \mathcal{D}(0, 1, 1)), \dim \mathcal{D}(1, 1, -1), \dim \mathcal{D}(0, 1, -1)].\end{aligned}$$

Table 1
Indecomposable three-dimensional complex Lie algebras and their invariant function ψ

$\mathcal{L}$	Commutators	inv($\mathcal{L}$)	Function ψ				
$A_{3,1}$	$[e_2, e_3] = e_1$	(310)(310)(13) 1	α				
		[6, 6, 3, 5, 3, 4]	$\psi(\alpha)$	6			
$A_{3,2}$	$[e_1, e_3] = e_1$,	(320)(32)(0) 1	α	1			
	$[e_2, e_3] = e_1 + e_2$	[4, 3, 1, 2, 0, 1]	$\psi(\alpha)$	4	3		
$A_{3,3}$	$[e_1, e_3] = e_1$,	(320)(32)(0) 1	α	1			
	$[e_2, e_3] = e_2$	[6, 3, 1, 2, 0, 1]	$\psi(\alpha)$	6	3		
$A_{3,4}$	$[e_1, e_3] = e_1$,	(320)(32)(0) 1	α	-1	1		
	$[e_2, e_3] = -e_2$	[4, 3, 1, 2, 0, 1]	$\psi(\alpha)$	5	4	3	
$A_{3,5}(a)$	$[e_1, e_3] = e_1$,	(320)(32)(0) 1	α	1	a	$\frac{1}{a}$	
	$[e_2, e_3] = ae_2$, $0 < \lvert a\rvert < 1$ or $\lvert a\rvert = 1$, $\operatorname{Im} a > 0$	[4, 3, 1, 2, 0, 1]	$\psi(\alpha)$	4	4	4	3
$A_{3,8}$	$[e_1, e_3] = -2e_2$,	(3)(3)(0) 1	α	-1	1	2	
	$[e_1, e_2] = e_1$, $[e_2, e_3] = e_3$	[3, 0, 1, 0, 0, 1]	$\psi(\alpha)$	5	3	1	0

Blank space in the smaller table for function ψ denotes a general complex number, different from all previously listed values, e.g. for $A_{3,8}$ it holds that $\psi(\alpha) = 0, \alpha \neq -1, 1, 2$.

3.2. Application of the invariant function ψ to three-dimensional Lie algebras

Since among three-dimensional Lie algebras an infinite continuum already appears, it is clear that the finite set inv($\mathcal{L}$) of certain dimensions, though useful, can never completely characterize Lie algebras of dimension higher than or equal to 3. On the contrary, it turns out that the invariant function ψ alone(!) forms a complete set of invariant(s) for indecomposable three-dimensional Lie algebras, as Table 1 shows. We use the notation for three-dimensional Lie algebras as in [11]. We point out the case of $A_{3,5}(a)$, where the function ψ is different for different values of the parameter $a \in \mathbb{C}$, $0 < |a| < 1$ or $|a| = 1$, $\operatorname{Im} a > 0$, and thus distinguishes among non-isomorphic algebras in this continuum. One may also find convenient that once one has some structure constants of any indecomposable three-dimensional Lie algebra, simple computation of function ψ allows an unambiguous identification in the list (see also Section 4). We may also mention here that the invariant function ϕ has for $A_{3,1}$ a single value $\phi(\alpha) = 3$, and for the remaining algebras $A_{3,i}$, $i = 2, \ldots, 8$, it holds that

$$\phi(\alpha) = \begin{cases} 1, & \alpha = 1 \\ 0, & \alpha \neq 1. \end{cases}$$

3.3. Application of the function ψ to some eight-dimensional Lie algebras

Example 3.1. Let us introduce two eight-dimensional complex Lie algebras $\mathcal{L}$, $\widetilde{\mathcal{L}}$ by listing their commutation relations in the basis $\{e_1, \ldots, e_8\}$:

$$\begin{aligned} \mathcal{L} \quad & [e_1, e_3] = e_5,\ [e_1, e_4] = e_8,\ [e_1, e_5] = e_7,\ [e_1, e_6] = e_4, \\ & [e_2, e_3] = e_7,\ [e_3, e_5] = e_8,\ [e_4, e_6] = e_7 \\ \widetilde{\mathcal{L}} \quad & [e_1, e_3] = e_5,\ [e_1, e_4] = e_8,\ [e_1, e_6] = e_4,\ [e_2, e_3] = e_7, \\ & [e_2, e_6] = e_8,\ [e_3, e_5] = e_8,\ [e_4, e_6] = e_7. \end{aligned}$$

These algebras are both indecomposable and nilpotent. They are both the result of a contraction of $sl(3, \mathbb{C})$ and form so called *continuous graded contractions* corresponding to the Pauli grading of $sl(3, \mathbb{C})$ [12,6]. They appear on the list in [6] named as $\mathcal{L}_{17,9}$ and $\mathcal{L}_{17,12}$. Computing their invariants we obtain

$$\begin{aligned} &\text{inv}(\mathcal{L}) \quad (8, 4, 0)(8, 4, 2, 0)(2, 5, 8) \quad 2 \quad [16, 19, 9, 11, 8, 17] \\ &\text{inv}(\widetilde{\mathcal{L}}) \quad (8, 4, 0)(8, 4, 2, 0)(2, 5, 8) \quad 2 \quad [16, 19, 9, 11, 8, 17]. \end{aligned}$$

Here we observe that a unique characterization is still not attained. On the contrary, computing invariant functions ϕ, ψ and $\widetilde{\phi}$, $\widetilde{\psi}$ for algebras $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ yields:

α	0	-2	
$\psi(\alpha)$	19	17	16

α	0	1	
$\phi(\alpha)$	16	9	8
$\widetilde{\phi}(\alpha)$	16	9	8

α	0	$-\frac{1}{2}$	
$\widetilde{\psi}(\alpha)$	19	17	16

Since $\psi \neq \widetilde{\psi}$, we conclude that $\mathcal{L} \ncong \widetilde{\mathcal{L}}$.

Example 3.2. We present Lie brackets for the indecomposable eight-dimensional nilpotent one-parametric Lie algebra:

$$\begin{aligned}\mathcal{L}(a)\quad & [e_1,e_3]=e_5,\ [e_1,e_4]=-ae_8,\ [e_2,e_3]=e_7,\ [e_2,e_4]=e_6,\\ & [e_3,e_5]=e_8,\ [e_3,e_7]=e_6,\ 0\leq|a|\leq 1.\end{aligned}$$

This continuum appeared as $\mathcal{L}_{18,25}(a)$ in [6]; however, the relations among its algebras remained unresolved there. We achieve partial characterization by isolating two of its points, $a=0,-1$, and thus obtain

$\mathrm{inv}(\mathcal{L}(0))$		$(8,4,0)(8,4,2,0)(2,5,8)$	4	$[21,23,10,14,9,18]$
$\mathrm{inv}(\mathcal{L}(-1))$		$(8,4,0)(8,4,2,0)(2,5,8)$	4	$[22,22,10,13,9,18]$
$\mathrm{inv}(\mathcal{L}(a))$	$a\neq 0,-1$	$(8,4,0)(8,4,2,0)(2,5,8)$	4	$[20,22,10,13,9,18]$.

We summarize the tables of invariant functions ϕ_a and ψ_a of $\mathcal{L}(a)$ as follows:

α	0	1	
$\phi_a(\alpha)$	16	10	9

$a=0$			$a=1$					$a=-1$				$a\neq 0,\pm 1$					
α	0		α	0	-1	1		α	0	1		α	0	1	$-a$	$-\frac{1}{a}$	
$\psi_0(\alpha)$	23	21	$\psi_1(\alpha)$	22	21	20	19	$\psi_{-1}(\alpha)$	22	22	19	$\psi_a(\alpha)$	22	20	20	20	19

Finally, the invariant function ψ_a provides us with a complete characterization of the continuum; or more precisely: since for $a,b\neq 0,\pm 1,\ a\neq b,\ a\neq\frac{1}{b}$, it holds that

$$\psi_a(-a)=20\neq\psi_b(-a)=19,$$

the relation $\mathcal{L}(a)\ncong\mathcal{L}(b)$ is thus guaranteed.

Example 3.3. Like in the previous example, we list commutators of the indecomposable eight-dimensional nilpotent one-parametric Lie algebra:

$$\begin{aligned}\mathcal{L}(a)\quad & [e_1,e_3]=ae_5,\ [e_1,e_4]=e_8,\ [e_1,e_6]=e_4,\ [e_2,e_3]=e_7,\\ & [e_3,e_5]=e_8,\ [e_3,e_6]=e_2,\ [e_4,e_6]=e_7,\ 0<|a|\leq 1.\end{aligned}$$

This class of Lie algebras also appeared named as $\mathcal{L}_{17,13}(a)$ in [6] and the relations among its algebras remained also unresolved. Isolating one of its points, $a=-1$, we obtain

$\mathrm{inv}(\mathcal{L}(-1))$		$(8,5,0)(8,5,2,0)(2,5,8)$	4	$[19,19,8,9,7,18]$
$\mathrm{inv}(\mathcal{L}(a))$	$a\neq -1$	$(8,5,0)(8,5,2,0)(2,5,8)$	4	$[17,19,8,9,7,18]$.

We list the invariant functions ϕ_a and ψ_a for $\mathcal{L}(a)$:

α	0	1	
$\phi_a(\alpha)$	16	8	7

$a=1$					$a=-1$					$a\neq\pm1$						
α	-1	0	1		α	0	1	-1		α	0	1	-1	$-a$	$-\frac{1}{a}$	
$\psi_1(\alpha)$	19	19	17	16	$\psi_{-1}(\alpha)$	19	19	17	16	$\psi_a(\alpha)$	19	17	17	17	17	16

which enables us to conclude that the function ψ_a again represents a priceless instrument providing a complete description of the presented parametric continuum of Lie algebras.

4. Concluding remarks

- There are several non-equivalent ways of generalizing the notion of derivation of Lie algebra. For example in [5], a linear operator $A \in \mathrm{End}(\mathcal{L})$ is called a (σ, τ)-derivation of $\mathcal{L}$ if for some $\sigma, \tau \in \mathrm{End}(\mathcal{L})$ and all $x, y \in \mathcal{L}$,

  $$A[x, y] = [Ax, \tau y] + [\sigma x, Ay].$$

  This generalization for the σ, τ homomorphisms appears already in [7]. If there exists $B \in \mathrm{der}(\mathcal{L})$ such that for all $x, y \in \mathcal{L}$ the condition $A[x, y] = [Ax, y] + [x, By]$ holds, then the operator A forms another generalization [2]. A more general definition emerged in [8] and runs as follows: $A \in \mathrm{End}(\mathcal{L})$ is called the *generalized derivation* of $\mathcal{L}$ if there exist $B, C \in \mathrm{End}(\mathcal{L})$ such that for all $x, y \in \mathcal{L}$ the property $C[x, y] = [Ax, y] + [x, By]$ holds.
- The sets $\mathcal{D}(1, 1, 0)$ and $\mathcal{D}(0, 1, -1)$ are called *centroid* and *quasicentroid* respectively in [8]. The question of under which conditions these sets coincide has also been discussed.
- Jordan algebra $\mathcal{D}(1, 1, -1)$, together with Lie algebras $\mathcal{D}(0, 1, 1)$, $\mathcal{D}(1, 0, 0) \cap \mathcal{D}(0, 1, 1)$, still deserves further study as regards structure and mutual relations.
- Having a matrix $A = (A_{ij})$ and structure constants c_{ij}^k of $\mathcal{L}$ in some basis, then in order to have $A \in \mathcal{D}(\alpha, \beta, \gamma)$ one has to solve the system of linear homogeneous equations

  $$\sum_m (\alpha c_{ij}^m A_{mk} + \beta c_{mj}^k A_{mi} + \gamma c_{im}^k A_{mj}) = 0, \quad i, j, k = 1, 2, \ldots, \dim \mathcal{L}.$$

  This shows how the computation of spaces $\mathcal{D}(\alpha, \beta, \gamma)$ (in fact, due to Theorem 2.2 one parameter is sufficient) and consequently the function ψ is comparatively very easy. This computation is also viable in higher dimensions — as presented in Section 3.3.
- Compared to the extensive usefulness of the function ψ, we haven't found much use for the invariant function ϕ; the form of the function ϕ is, however, non-trivial and its general behaviour represents an open problem.
- The notion of (α, β, γ)-derivations and a theorem similar to Theorem 2.2 can be derived for general commutative or anti-commutative algebra.
- As expected, though function ψ forms a complete invariant in dimension 3, this is no longer true in dimension 4 (but it still works nicely there), let alone in higher dimensions. It is likely that some characteristics similar to the invariant functions presented, more general perhaps, could complete the scenery of invariants. One can only encourage such a pursuit as it seems to be the right way to go.

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